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2005 J. Phys. A: Math. Gen. 38 9969

(http://iopscience.iop.org/0305-4470/38/46/005)

J. Phys. A: Math. Gen. **38** (2005) 9969–9978 doi:10.1088/0305-4470/38/46/005

Asymptotics of the information entropy of the Airy function

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Received 31 May 2005, in final form 30 August 2005
Published 2 November 2005
Online at stacks.iop.org/JPhysA/38/9969

Abstract
The Boltzmann–Shannon information entropy of linear potential wavefunctions is known to be controlled by the information entropy of the Airy function Ai(x). Here, the entropy asymptotics is analysed so that the first two leading terms (previously calculated in the WKB approximation) as well as the following term (already conjectured) are derived by using only the specific properties of the Airy function.

PACS numbers: 02.30.Gp, 02.30.Mv, 03.65.Ge
Mathematics Subject Classification: 33C10, 41A60, 94A17

1. Introduction

The Boltzmann–Shannon information entropy of an individual microstate of a one-dimensional physical system is defined by

$$S[\Psi] := -\int |\Psi(x)|^2 \log|\Psi(x)|^2 \, \mathrm{d}x, \tag{1}$$

where $\Psi(x)$ is the quantum-mechanical wavefunction of the state [1, 2]. The exact computation of this quantity is a formidable, practically impossible task. This is because either the Schrödinger equation of the system cannot be exactly solved or, when it is, the wavefunction is a known special function whose information entropy cannot be analytically evaluated and its numerical computation is highly unstable. Most efforts have been focused on classical hypergeometric functions of the type $\Psi(x) = \sqrt{\omega(x)}y_n(x)$, where $\{y_n(x); n = 0, 1, \ldots\}$ denotes a set of polynomials orthogonal with respect to the weight function $\omega(x)$. In this case, the computation of the entropy (1) boils down to evaluating the so-called information entropy of the orthogonal polynomials $y_n(x)$, given by

$$S[y_n] := -\int y_n^2(x)\big[\log y_n^2(x)\big]\omega(x)\,\mathrm{d}x.$$

Briefly, up to now, this mathematical quantity has been exactly calculated only for Chebyshev polynomials and some Gegenbauer polynomials with an integer parameter, although numerous results for other classical orthogonal polynomials (Hermite, Laguerre, Jacobi) have been found, mainly from an asymptotical point of view. See [3] for a recent summary of the main achievements. The numerical computation of this entropic integral on finite intervals is most conveniently done by the effective method of Buyarov *et al* [4].

Recently, the evaluation of the physical entropy (1) has been attacked for wavefunctions controlled by special functions other than orthogonal polynomials. This occurs for systems such as the circular membrane [5], the Toda-like potential [6] and confining power-type potential [7]. For these cases the entropy has been evaluated in the asymptotic region, that is, for highly energetic quantum states, either by using only the properties of the involved special functions or by means of the semiclassical or WKB approximation. In doing so, the asymptotics of the information entropies of the Bessel functions [5] and the McDonald functions [6] have been calculated.

In this paper, we shall consider the computation of the Boltzmann–Shannon entropy for a linear potential, i.e., for the quantum-mechanical problem of a particle under the influence of a constant (non-vanishing) force. This potential for a vanishing orbital quantum number and the harmonic oscillator potential are the only two power-type confinement potentials [8, 9], so much used with quark models and more particularly with the charmonium model [10] of the J/ψ particle [11, 12], which are exactly solvable in terms of known special functions. The linear potential has the same importance in particle physics as the Coulomb potential in atomic physics and the harmonic potential in solid state physics. In section 2 it is pointed out that the linear potential wavefunctions for a vanishing orbital quantum number are controlled by the convergent solutions of Airy's equation, i.e., by the Airy function $\mathrm{Ai}(z)$. So, the calculation of the physical entropy of the highly energetic states of a particle in the linear potential boils down to the asymptotics of the information entropy of the Airy function.

The main purpose of this paper is the analytical derivation of the first three terms of this asymptotics by means of the specific properties of the Airy functions. The structure of this paper is as follows. In section 3, the asymptotics of the entropy of the Airy function S_n^{Ai} is investigated in detail by using its specific properties. Finally, a brief summary and some open problems are pointed out.

2. Linear potential wavefunctions and the Airy function: information entropies

The three-dimensional motion of a particle under a linear potential for the case of a vanishing orbital quantum number reduces to the one-dimensional problem of the potential $V(x) = F|x|$, where F denotes the field strength [14, 15]. The wavefunctions of the quantum-mechanical states of the particle in such a potential are the solutions of the associated Schrödinger equation,

$$-\frac{1}{2}\frac{\mathrm{d}^2\Psi(x)}{\mathrm{d}x^2} + F|x|\Psi(x) = E\Psi(x),$$

where units $\hbar = m = 1$ have been used. Various authors (see, e.g., [14, 15]) have shown that the eigenfunctions can be expressed in terms of the Airy function as

$$\Psi_n(x) = \begin{cases} N_n \mathrm{Ai}(\alpha|x| + \beta_n), & \text{even } n \\ \mathrm{sign}(x) N_n \mathrm{Ai}(\alpha|x| + \beta_n), & \text{odd } n, \end{cases} \tag{2}$$

where the normalization constant N_n and the parameters β_n and α are given by

$$\beta_n = \begin{cases} a'_{n/2+1} & \text{even } n \\ a_{(n+1)/2} & \text{odd } n, \end{cases}$$

$$N_n = \begin{cases} \sqrt{\dfrac{\alpha}{2}} \dfrac{1}{\sqrt{-\beta_n}\mathrm{Ai}(\beta_n)} & \text{even } n \\ \sqrt{\dfrac{\alpha}{2}} \dfrac{1}{\mathrm{Ai}'(\beta_n)} & \text{odd } n, \end{cases}$$

$$\alpha = (2F)^{1/3},$$

where a_s and a'_s are the zeros of the Airy function $\mathrm{Ai}(x)$ and its derivative $\mathrm{Ai}'(x)$ respectively.

The associated energy eigenvalues are

$$E_n = -\frac{F}{\alpha}\beta_n.$$

The spreading of the linear potential wavefunctions, $\Psi_n(x)$, is best described [1] by the Boltzmann–Shannon information entropy (1), which gives the spatial distribution of the associated quantum-mechanical Born density $|\Psi_n(x,t)|^2 = |\Psi_n(x)|^2$. Moreover, it gives the uncertainty of the position of the particle. Taking into account equation (2), this entropy is expressed as

$$S[\Psi_n] = -\int_{-\infty}^{\infty} |\Psi_n(x)|^2 \log |\Psi_n(x)|^2 \, \mathrm{d}x = -\log N_n^2 + 2\frac{N_n^2}{\alpha} S_n^{\mathrm{Ai}}, \tag{3}$$

where S_n^{Ai} is the information entropy of the Airy function on the interval $[\beta_n, \infty)$:

$$S_n^{\mathrm{Ai}} = -\int_{\beta_n}^{\infty} \mathrm{Ai}^2(x) \log \mathrm{Ai}^2(x) \, \mathrm{d}x.$$

3. Entropy asymptotics

Taking into account the asymptotic expansion for the Airy function [16], we have that, when $x \ll 0$,

$$|\mathrm{Ai}(x) - \overline{\mathrm{Ai}}(x)| = O(x^{-7/4}),$$

where

$$\overline{\mathrm{Ai}}(x) = \frac{(-x)^{-1/4}}{\sqrt{\pi}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right).$$

So we can write

$$\begin{aligned} S_n^{\mathrm{Ai}} &= -\int_{\beta_n}^{\infty} \mathrm{Ai}^2(x) \ln \mathrm{Ai}^2(x) \, \mathrm{d}x \\ &= -\int_{\beta_n}^{0} \mathrm{Ai}^2(x) \ln \mathrm{Ai}^2(x) \, \mathrm{d}x - \int_{0}^{\infty} \mathrm{Ai}^2(x) \ln \mathrm{Ai}^2(x) \, \mathrm{d}x \\ &= -\int_{\beta_n}^{0} (\mathrm{Ai}^2(x) \ln \mathrm{Ai}^2(x) - \overline{\mathrm{Ai}}^2(x) \ln \overline{\mathrm{Ai}}^2(x)) \, \mathrm{d}x \\ &\quad - \int_{\beta_n}^{0} \overline{\mathrm{Ai}}^2(x) \ln \overline{\mathrm{Ai}}^2(x) \, \mathrm{d}x - \int_{0}^{\infty} \mathrm{Ai}^2(x) \ln \mathrm{Ai}^2(x) \, \mathrm{d}x \end{aligned}$$

$$= -\int_{-\infty}^{0} (\mathrm{Ai}^2(x)\ln\mathrm{Ai}^2(x) - \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x))\,\mathrm{d}x$$
$$+ \int_{-\infty}^{\beta_n} (\mathrm{Ai}^2(x)\ln\mathrm{Ai}^2(x) - \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x))\,\mathrm{d}x$$
$$- \int_{\beta_n}^{0} \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x)\,\mathrm{d}x - \int_0^\infty \mathrm{Ai}^2(x)\ln\mathrm{Ai}^2(x)\,\mathrm{d}x$$
$$= -\int_{\beta_n}^{0} \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x)\,\mathrm{d}x + \kappa + o(1),$$

where, since β_n tends to $-\infty$ as n increases,

$$\int_{-\infty}^{\beta_n} (\mathrm{Ai}^2(x)\ln\mathrm{Ai}^2(x) - \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x))\,\mathrm{d}x = o(1),$$

and

$$\kappa = -\int_0^\infty \mathrm{Ai}^2(x)\ln\mathrm{Ai}^2(x)\,\mathrm{d}x - \int_{-\infty}^{0} (\mathrm{Ai}^2(x)\ln\mathrm{Ai}^2(x) - \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x))\,\mathrm{d}x$$
$$\simeq 0.2265,$$

is a well-defined constant.

Then,

$$S_n^{\mathrm{Ai}} \simeq -\int_{\beta_n}^{0} \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x)\,\mathrm{d}x + \kappa. \tag{4}$$

Using the L^q-norm method (see, e.g., [17, 18]) we can write

$$-\int_{\beta_n}^{0} \overline{\mathrm{Ai}}^2(x)\ln\overline{\mathrm{Ai}}^2(x)\,\mathrm{d}x = -\left[\frac{\mathrm{d}}{\mathrm{d}q}\int_{\beta_n}^{0} (\overline{\mathrm{Ai}}^2(x))^q\,\mathrm{d}x\right]_{q=1}. \tag{5}$$

Considering the asymptotics of the zeros β_n [16],

$$a_s = -T\left(\frac{3}{8}\pi(4s-1)\right), \qquad T(t) \sim t^{2/3},$$
$$a'_s = -U\left(\frac{3}{8}\pi(4s-3)\right), \qquad U(t) \sim t^{2/3},$$

and the change of variable, $x = \beta_n(y/\pi)^{2/3}$, we have

$$\int_{\beta_n}^{0} (\overline{\mathrm{Ai}}^2(x))^q\,\mathrm{d}x = \frac{1}{\pi^q}\frac{2}{3}\left(\frac{3}{8}(2n+1)\right)^{\frac{2-q}{3}}$$
$$\times\left[\int_0^\pi \left[\sin^2\left(\frac{1}{2}ny + \frac{1}{4}(y+\pi)\right)\right]^q y^{-\frac{q+1}{3}}\,\mathrm{d}y\right] + O(n^{-1})$$
$$= \frac{1}{\pi^q}\frac{2}{3}\left(\frac{3}{4}n\right)^{\frac{2-q}{3}}\left[\frac{3}{2-q}\pi^{\frac{1-2q}{6}}\frac{\Gamma(q+\frac{1}{2})}{\Gamma(q+1)} + \tau(q)n^{\frac{q-2}{3}}\right] + o(1), \tag{6}$$

with

$$\tau(q) = 2^{\frac{2-q}{3}}\left[\int_{\frac{\pi}{4}}^{\pi}\left(x-\frac{\pi}{4}\right)^{-\frac{1+q}{3}}\sin^{2q}x\,\mathrm{d}x - \frac{3}{(2-q)}\pi^{-1}\int_0^\pi\left(x+\frac{\pi}{4}\right)^{\frac{2-q}{3}}\sin^{2q}x\,\mathrm{d}x\right.$$
$$\left.-\frac{1+q}{3}\pi\int_{\frac{1}{2}}^{\frac{3}{2}}\left(\{t\}-\frac{1}{2}\right)\int_0^\infty \frac{\sin^{2q}x\,\mathrm{d}x}{\left(x+\pi t-\frac{\pi}{4}\right)^{\frac{4+q}{3}}}\,\mathrm{d}t\right],$$

where the first summand in equation (6) and the expression of $\tau(q)$ are proved in the appendix. Note that $\{t\}$ is the fractional part of t.

Thus, according to equations (5) and (6), differentiating with respect to q and taking $q = 1$, we obtain the additional constant κ' in the limit

$$-\int_{\beta_n}^{0} \overline{\mathrm{Ai}}^2(x) \ln \overline{\mathrm{Ai}}^2(x)\,\mathrm{d}x = C\left[\frac{1}{3}n^{\frac{1}{3}}\ln n + Dn^{\frac{1}{3}}\right] + \kappa' + o(1), \tag{7}$$

where $C = \frac{1}{\pi}\left(\frac{3}{4}\pi\right)^{\frac{1}{3}}$, $D = \frac{1}{3}\ln(48\pi^4) - 2$, and

$$\kappa' = -\frac{1}{3\pi}\left(\frac{2}{9}\right)^{\frac{1}{3}}\left[\left(\log\frac{4}{3} - 3\log\pi\right)\tau(1) + 3\tau'(1)\right].$$

The computation of κ' leads us to the value $\kappa' = 0.1519$.

Equations (4) and (7) provide the following asymptotical behaviour of the entropy of the Airy function,

$$S_n^{\mathrm{Ai}} = C\left[\frac{1}{3}n^{\frac{1}{3}}\ln n + Dn^{\frac{1}{3}}\right] + \kappa + \kappa' + o(1).$$

For completeness, let us write down that the Boltzmann–Shannon information entropy (3) of the linear potential wavefunctions for the highly excited particle state is consequently given by

$$S[\Psi_n] = \frac{2}{3}\ln n + \left(\ln\left(\frac{2(6\pi)^{2/3}}{\alpha}\right) - 2\right) + \frac{\kappa+\kappa'}{C}n^{-\frac{1}{3}} + o(n^{-\frac{1}{3}}).$$

And the numerical value of the coefficient of $n^{-\frac{1}{3}}$ is $K = \frac{\kappa+\kappa'}{C} = 0.8934$.

The leading terms were previously computed within the framework of the WKB approximation by Sánchez-Ruiz [13]. Moreover, the same author, on the basis of some numerical experiments, also conjectured the following term $K'n'^{-1/3}$ with $K' = 0.709$ by taking into account only the odd-parity eigenfunctions; one can easily find with the same method that $K = 0.8933$ when both odd-parity and even-parity levels are considered. Note that if $n' = 1, 2, 3, \ldots$ is running only over odd states, and $n = 1, 2, 3, \ldots$ is running over even and odd states, $K'n'^{-1/3} = Kn^{-1/3} = K(2n'+1)^{-1/3} \simeq K2^{-1/3}n'^{-1/3}$, so $K = 2^{1/3}K'$, as can be seen from the previous values.

The importance of our work is to have determined and obtained analytically the first three asymptotical terms for the information entropy of the linear potential by means of only the specific properties of the involved Airy function.

4. Summary and open problems

We have determined the asymptotics of the information entropy of the Airy function $\mathrm{Ai}(x)$, whose usefulness for the computation of the Boltzmann–Shannon entropy of the highly energetic (Rydberg) quantum-mechanical states of the linear potential with vanishing angular momenta is also shown. In doing so, the only existing related results [13], which are based on the WKB approximation, are rigorously corroborated.

Let us point out, for completeness, that the physical entropies of the non-vanishing angular momentum Rydberg states of the linear potential [19–21], so useful for the theory of quark confinement, require previous knowledge of the information entropies of the first-order derivatives of the Airy function $\mathrm{Ai}'(x)$ as well as the combination of $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$. These two problems have not yet been solved.

Acknowledgments

We are grateful to A Aptekarev, J S Dehesa and A Martínez-Finkelshtein for discussions and suggestions. This work has been partially supported by the MCYT project no FIS2005-00973, as well as by the NATO collaborative grant PST.CLG.929738. We (PSM and RJY) belong to the Group FQM-207 of the Junta de Andalucía (Spain).

Appendix

We start with the integral

$$I_q = \int_0^\pi x^{-\frac{1+q}{3}} \left(\sin^2\left(\frac{2n+1}{4}x + \frac{\pi}{4}\right)\right)^q \,\mathrm{d}x.$$

The change of variable $y = x + \frac{\pi}{2n+1}$, and the notation $\gamma = \frac{1+q}{3}$ allow us to write that

$$I_q = \int_{\frac{\pi}{2n+1}}^{\pi+\frac{\pi}{2n+1}} \left(y - \frac{\pi}{2n+1}\right)^{-\gamma} \left(\sin^2\left(\frac{2n+1}{4}y\right)\right)^q \,\mathrm{d}y.$$

Now, we make the following partition of the interval of integration:

$$P = \left\{\frac{\pi}{2n+1}, \frac{4\pi}{2n+1}, \ldots, \frac{4k\pi}{2n+1}, \ldots, \pi + \frac{\pi}{2n+1}\right\}.$$

Thus, by integrating in every sub-interval separately and taking into account the periodicity of the sine function,

$$\begin{aligned} I_q = &\int_{\frac{\pi}{2n+1}}^{\frac{4\pi}{2n+1}} \left(y - \frac{\pi}{2n+1}\right)^{-\gamma} \sin^{2q}\left(\frac{2n+1}{4}y\right) \mathrm{d}y \\ &+ \sum_{k=1}^{[\frac{n+1}{2}]-1} \int_{\frac{4k\pi}{2n+1}}^{\frac{4(k+1)\pi}{2n+1}} \left(y - \frac{\pi}{2n+1}\right)^{-\gamma} \sin^{2q}\left(\frac{2n+1}{4}\left(y - \frac{4k\pi}{2n+1}\right)\right) \mathrm{d}y \\ &+ \int_{4[\frac{n+1}{2}]\frac{\pi}{2n+1}}^{\pi+\frac{\pi}{2n+1}} \left(y - \frac{\pi}{2n+1}\right)^{-\gamma} \sin^{2q}\left(\frac{2n+1}{4}\left(y - 4\left[\frac{n+1}{2}\right]\frac{\pi}{2n+1}\right)\right) \mathrm{d}y, \end{aligned}$$

with being the integer part of t.

With $[t]$ being the integer part of t now, we perform the following changes of variables in the three above integrals,

$$\begin{aligned} x &= \frac{2n+1}{4}y \\ x &= \frac{2n+1}{4}\left(y - \frac{4k\pi}{2n+1}\right) \\ x &= \frac{2n+1}{4}\left(y - 4\left[\frac{n+1}{2}\right]\frac{\pi}{2n+1}\right), \end{aligned}$$

respectively. Thus,

$$\begin{aligned} I_q = \left(\frac{4}{2n+1}\right)^{1-\gamma} \Bigg[&\int_{\frac{\pi}{4}}^{\pi} \left(x - \frac{\pi}{4}\right)^{-\gamma} \sin^{2q} x \,\mathrm{d}x \\ &+ \int_0^\pi \sum_{k=1}^{[\frac{n+1}{2}]-1} \left(x + k\pi - \frac{\pi}{4}\right)^{-\gamma} \sin^{2q} x \,\mathrm{d}x + Y_0 \Bigg], \end{aligned} \tag{A.1}$$

with

$$Y_0 = \int_0^{\frac{\pi}{4}(2n+2-4[\frac{n+1}{2}])} \left(x + \left[\frac{n+1}{2}\right]\pi - \frac{\pi}{4}\right)^{-\gamma} \sin^{2q} x \, \mathrm{d}x,$$

where $\left(x + \left[\frac{n+1}{2}\right]\pi - \frac{\pi}{4}\right)^{-\gamma} \to 0$ as n increases, so $Y_0 = o(1)$.

In order to continue we need the following lemma:

Lemma.

$$\sum_{\frac{1}{2}<k\leqslant n+\frac{1}{2}} F(k) = \int_{\frac{1}{2}}^{n+\frac{1}{2}} F(t)\,\mathrm{d}t + \int_{\frac{1}{2}}^{n+\frac{1}{2}} F'(t)B(t)\,\mathrm{d}t,$$

where $B(t) = \{t\} - \frac{1}{2}$.

Proof. Taking the values $a = \frac{1}{2}, b = n + \frac{1}{2}, m = 0$ into the Euler–McLaurin summation formula,

$$\sum_{a<k\leqslant b} F(k) = \int_a^b F(t)\,\mathrm{d}t + \sum_{r=0}^{m} \frac{(-1)^{r+1}}{(r+1)!}\left(B_{r+1}(b)F^{(r)}(b) - B_{r+1}(a)F^{(r)}(a)\right)$$
$$+ \frac{(-1)^m}{(m+1)!}\int_a^b B_{m+1}(t)F^{(k+1)}(t)\,\mathrm{d}t$$

and since the first periodic Bernoulli function is $B_1(t) = \{t\} - \frac{1}{2}$, this expression reduces to that of the lemma. □

Now, we apply this lemma to the sum involved in equation (A.1),

$$\sum_{k=1}^{[\frac{n+1}{2}]-1} \left(x + k\pi - \frac{\pi}{4}\right)^{-\gamma} = \int_{\frac{1}{2}}^{[\frac{n+1}{2}]-1} \left(x + t\pi - \frac{\pi}{4}\right)^{-\gamma} \mathrm{d}t$$
$$- \pi\gamma \int_{\frac{1}{2}}^{[\frac{n+1}{2}]-1} \left(x + t\pi - \frac{\pi}{4}\right)^{-\gamma-1} B(t)\,\mathrm{d}t$$
$$= \frac{\pi^{-1}}{1-\gamma}\left(x + t\pi - \frac{\pi}{4}\right)^{-\gamma+1}\Bigg|_{\frac{1}{2}}^{[\frac{n+1}{2}]-1}$$
$$- \pi\gamma \int_{\frac{1}{2}}^{\infty} \left(x + t\pi - \frac{\pi}{4}\right)^{-\gamma-1} B(t)\,\mathrm{d}t + Y_1,$$

where

$$Y_1 = \pi\gamma \int_{[\frac{n+1}{2}]-1}^{\infty} \left(x + \pi t - \frac{\pi}{4}\right)^{-\gamma-1} B(t)\,\mathrm{d}t.$$

Note that, as the integrand decreases with t and the interval of integration diminishes with n, we have $Y_1 = o(1)$.

Then,

$$\sum_{k=1}^{[\frac{n+1}{2}]-1} \left(x + k\pi - \frac{\pi}{4}\right)^{-\gamma} = \frac{\pi^{-\gamma}}{1-\gamma}\left(\frac{2n+1}{4}\right)^{-\gamma+1}\left(1 + \frac{x - \frac{\pi}{2} - \pi\left\{\frac{n+1}{2}\right\}}{\pi\,\frac{2n+1}{4}}\right)^{-\gamma+1}$$
$$- \frac{\pi^{-1}}{1-\gamma}\left(x + \frac{\pi}{4}\right)^{-\gamma+1} - \pi\gamma\int_{\frac{1}{2}}^{\infty}\left(x + \pi t - \frac{\pi}{4}\right)^{-\gamma-1} B(t)\,\mathrm{d}t + Y_1.$$

Taking into account the periodicity of $B(t)$, we can write

$$\sum_{k=1}^{[\frac{n+1}{2}]-1}\left(x+k\pi-\frac{\pi}{4}\right)^{-\gamma}=\frac{\pi^{-\gamma}}{1-\gamma}\left(\frac{2n+1}{4}\right)^{-\gamma+1}\left(1+\frac{x-\frac{\pi}{2}-\pi\left\{\frac{n+1}{2}\right\}}{\pi\,\frac{2n+1}{4}}\right)^{-\gamma+1}$$
$$-\frac{\pi^{-1}}{1-\gamma}\left(x+\frac{\pi}{4}\right)^{-\gamma+1}-\pi\gamma\int_{\frac{1}{2}}^{\frac{3}{2}}\sum_{k=0}^{\infty}\left(x+\pi(t+k)-\frac{\pi}{4}\right)^{-\gamma-1}B(t)\,\mathrm{d}t+Y_1.$$

Then, our integral in equation (A.1) becomes

$$\left(\frac{4}{2n+1}\right)^{-\gamma+1}\int_0^{\pi}\sum_{k=1}^{[\frac{n+1}{2}]-1}\left(x+k\pi-\frac{\pi}{4}\right)^{-\gamma}\sin^{2q}x\,\mathrm{d}x$$
$$=\frac{\pi^{-\gamma}}{1-\gamma}\int_0^{\pi}\left(1+\frac{x-\frac{\pi}{2}-\pi\left\{\frac{n+1}{2}\right\}}{\pi\,\frac{2n+1}{4}}\right)^{-\gamma+1}\sin^{2q}x\,\mathrm{d}x$$
$$-\left(\frac{4}{2n+1}\right)^{-\gamma+1}\frac{\pi^{-1}}{1-\gamma}\int_0^{\pi}\left(x+\frac{\pi}{4}\right)^{-\gamma+1}\sin^{2q}x\,\mathrm{d}x$$
$$-\left(\frac{4}{2n+1}\right)^{-\gamma+1}\pi\gamma\int_{\frac{1}{2}}^{\frac{3}{2}}B(t)\int_0^{\infty}\frac{\sin^{2q}x\,\mathrm{d}x}{\left(x+\pi t-\frac{\pi}{4}\right)^{\gamma+1}}\,\mathrm{d}t,$$

where we have taken into account the periodicity of $\sin^{2q}x$ and some interchanges of signs of integration and summatory have been done.

Working in the first integral in the second term of the previous expression,

$$\left(1+\frac{x-\frac{\pi}{2}-\pi\left\{\frac{n+1}{2}\right\}}{\pi\,\frac{2n+1}{4}}\right)^{-\gamma+1}=1+(1-\gamma)\frac{x-\frac{\pi}{2}-\pi\left\{\frac{n+1}{2}\right\}}{\pi\,\frac{2n+1}{4}}+\cdots,$$

we define

$$Y_2=\frac{\pi^{-\gamma}}{1-\gamma}\int_0^{\pi}\left(4(1-\gamma)\frac{x-\frac{\pi}{2}-\pi\left\{\frac{n+1}{2}\right\}}{(2n+1)\pi}+\cdots\right)\sin^{2q}x\,\mathrm{d}x.$$

Here, as the integrand goes as n^{-1}, $Y_2=o(1)$.

Thus,

$$I_q=\frac{\pi^{-\gamma}}{1-\gamma}\int_0^{\pi}\sin^{2q}x\,\mathrm{d}x+\left(\frac{4}{2n+1}\right)^{-\gamma+1}$$
$$\times\left[\int_{\frac{\pi}{4}}^{\pi}\left(x-\frac{\pi}{4}\right)^{-\gamma}\sin^{2q}x\,\mathrm{d}x-\frac{\pi^{-1}}{1-\gamma}\int_0^{\pi}\left(x+\frac{\pi}{4}\right)^{-\gamma+1}\sin^{2q}x\,\mathrm{d}x\right.$$
$$\left.-\pi\gamma\int_{\frac{1}{2}}^{\frac{3}{2}}B(t)\int_0^{\infty}\frac{\sin^{2q}x\,\mathrm{d}x}{\left(x+\pi t-\frac{\pi}{4}\right)^{\gamma+1}}\,\mathrm{d}t+\int_0^{\pi}Y_1\sin^{2q}x\,\mathrm{d}x\right]+Y_2+Y_0.$$

Finally, taking into account the expressions of γ, considering that Y_0, Y_1 and Y_2 are of order $o(1)$, and that

$$\int_0^{\pi} \sin^{2q} x \, dx = \frac{\sqrt{\pi}\,\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q+1)},$$

we have

$$I_q = \frac{3}{2-q}\pi^{\frac{1-2q}{6}} \frac{\Gamma\left(q+\frac{1}{2}\right)}{\Gamma(q+1)} + \left(\frac{4}{2n+1}\right)^{\frac{2-q}{3}} \times \left[\int_{\frac{\pi}{4}}^{\pi} \left(x-\frac{\pi}{4}\right)^{-\frac{1+q}{3}} \sin^{2q} x \, dx - \frac{3\pi^{-1}}{2-q} \int_0^{\pi} \left(x+\frac{\pi}{4}\right)^{\frac{2-q}{3}} \sin^{2q} x \, dx - \frac{1+q}{3}\pi \int_{\frac{1}{2}}^{\frac{3}{2}} B(t) \int_0^{\infty} \frac{\sin^{2q} x \, dx}{\left(x+\pi t-\frac{\pi}{4}\right)^{\frac{4+q}{3}}} \, dt \right] + o(1),$$

which was used in equation (6).

As a remark, the double integral in the last term of the previous expression can be done in a straightforward manner just by interchanging the integral signs and integrating first in t. However, the result is more complicated, involving several integrals with an infinite value, so that only the sum of them is finite.

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